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Entropy of Semiclassical States in Chaotic Cosmology

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By evolving the state (density matrix) of the Mixmaster universe backward towards the singularity, it is shown that there must be a decrease of the entropy of the model. This result supports the idea that the arrow of time cannot point in the direction of contraction of the universe.

The evolution of the Mixmaster universe [homogeneous, diagonal, Bianchi type IX model (Ryan, 1975)] toward the initial singularity can be described asymptotically by giving the probability $p(z, \Delta z)$ that a Kasner configuration be found in the interval $(z, z+\Delta z)$ (Belinskii *et al.*, 1970; Misner, 1969; Barrow, 1982). Although the model is two-dimensional, it is conveniently approximated by an infinite sequence $\{z_n\}$ of Kasner configurations each one specified by a one-dimensional *Kasner parameter* $z \in (0, 1)$. The Poincaré map of the model (Belinski *et al.*, 1970; Misner, 1969; Barrow, 1982), denoted here by T, is responsible for transitions $z_n \to z_{n+1}$, where the singularity is approached as $n \to \infty$. The measure preserved by T is $\mu(z) = [(1+z) \ln z]^{-1}$, giving $p(z, \Delta z) = \int_z^{z+\Delta z} \mu(z')\, dz'$. A question that naturally arises is how an initial distribution μ_0 relaxes to μ under the action of T. It has been shown (Szüsz, 1968), that, for a θ-Lipschitz distribution μ_0, successive iterations of the Poincaré map result in the distribution $\mu_n(z) = \mu(z) + O(q^n)$, where $0 < q < 1$ depends only on θ. We will investigate what kind of conclusions can be drawn when using the asymptotic form $O(q^n) = e^{-\lambda n z}$, $\lambda > 0$,[2] in the context of semiclassical quantum cosmology. Probabilistic interpretation requires that all distributions be normalized and we redefine $\mu_n(z) = N_n[\mu(z) + e^{-\lambda n z}]$, with N_n a normalization factor.

As discussed elsewhere, we consider an initial wave function $|z\rangle$ peaked around the Kasner configuration z [this can be a Gaussian wave packet (Ryan, 1972; Furusawa, 1986), or else we can construct coherent states on

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the classical orbit (Francisco and Pilati, 1985)]. The semiclassical evolution is then given by an operator U_T where $U_T|z\rangle = |Tz\rangle$. The effect we want to study can be made explicit by a few iterations of T and we do not need to evolve $|z\rangle$ indefinitely toward the singularity, which would invalidate the semiclassical approximation (after a few iterations the system is completely chaotic and probabilistic distributions must be used).

The *state* of the system (or *density matrix*) associated with $|z\rangle$ and $\mu(z)$ is defined by (Wehrl, 1978) $\rho = \int_0^1 |z\rangle\mu(z)\langle z|\, dz$. The initial distribution μ_0 yields the family $\rho_n = \int_0^1 |z\rangle\mu_n(z)\langle z|\, dz$ and this converges exponentially fast to μ as $n \to \infty$. Notice that (i) the evolution $\rho_n \to \rho_{n+1}$ is well defined, since the form of μ_n implies $U_T\rho_n U_T^* = \rho_{n+1}$, (ii) ρ is an equilibrium state: $U_T\rho U_T^* = \rho$. An interesting concept can now be introduced (Wehrl, 1978; Thirring, 1983), the *entropy* of the state ρ:

$$S(\rho) = -\mathrm{Tr}(\rho \ln \rho) \tag{1}$$

In dealing with a semiclassical approximation, (1) simplifies to $S(\rho) = -\int \mu(z) \ln \mu(z)\, dz$.

The entropy (1) is a measure of how mixed the state ρ is and it gives an estimate of the disorder of the system. In quantum mechanics it is associated with a quantitative measurement of the information that is inaccessible to the observer: for a pure state $S(\rho) = 0$, while mixed states have $S(\rho) > 0$. We want to use the entropies $S(\rho_n)$ and determine how the proximity of the singularity influences the state of the system. In order to obtain a meaningful result, our conclusions must apply to the largest possible class of initial distributions compatible with the system. An essential property of the Mixmaster model enables us to determine the class to which μ_0 must belong. Using Misner's (1969) approach, the system can be represented by a time-dependent Hamiltonian whose potential is formed by very steep walls having the shape of an expanding equilateral triangle. A particle bouncing inside reproduces the geometric properties of the gravitational field. One can show that the vertex of the triangle has the same speed as the particle and consequently during the evolution the system spends a long time aligned with the vertices of the triangle (the bounces in such cases take longer to occur). This fact is expressed by the distributions μ_n, μ being peaked toward the corners [in (0, 1) only the corner $z = 0$ is present]. Thus, μ_0 must belong to the class of strictly decreasing functions; this and the Lipschitzian property guarantees that μ_n will be generated from this class.

Let $\xi = \{A_i\}_{i=1}^N$ be a partition of the interval (0, 1) and introduce the following family of coarse-grained entropies for the states ρ_n:

$$S_\xi(\rho_n) = -\sum p_n(A_i) \ln p_n(A_i) \tag{2}$$

where $p_n(A_i) = \int_{A_i} \mu_n(z)\, dz$. Our decision to use the discrete sums (2) instead of continuous integrals is based on the fact that mainly such integrals are not necessarily finite and positive, so that it is not always possible to interpret them as average information, and finally statistical mechanics requires the introduction of coarse graining (Wehrl, 1978). The evolution $S_\xi(\rho_n) \to S_\xi(\rho_{n+1})$ will produce a variation $\Delta S_\xi(n) = S_\xi(\rho_{n+1}) - S_\xi(\rho_n)$ and next we determine the sign of (ΔS) for $n>1$ by using the following inequality (Wehrl, 1978). Suppose the positive numbers $\{\alpha_i\}_{i=1}^N$ and $\{\beta_i\}_{i=1}^N$ satisfy (i) $\alpha_i \geq \alpha_{i+1}$, $\beta_i \geq \beta_{i+1}$, (ii) $\sum_{i=1}^N \alpha_i = 1 = \sum_{i=1}^N \beta_i$, and (iii) $\sum_{i=1}^k \alpha_i \leq \sum_{i=1}^k \beta_i$, $1 \leq k < N$. Then

$$-\sum_{i=1}^{N} \alpha_i \ln \alpha_i \geq -\sum_{i=1}^{N} \beta_i \ln \beta_i \tag{3}$$

Label the elements of ξ from $z=0$ to $z=1$ in increasing order and take $\alpha_i = p_n(A_i)$ and $\beta_i = p_{n+1}(A_i)$. Since μ_n are decreasing, (i) and (ii) are satisfied. Also, from $\int \mu_n = 1$ and $\mu_n(0) < \mu_{n+1}(0)$, the graphs of μ_n and μ_{n+1} must cross; the monotonicity of these functions implies that the intersection is unique and (iii) must hold. Consequently, $\Delta S < 0$ and there is a decrease of entropy as the singularity is approached.

The effect just described refers to propagating the system backward in time and it favors the idea that one should associate a low entropy with the singularity instead of attaching to it high-entropy states (Hawking, 1978). This may be related to the loss of one parameter during the evolution: although the model is 2-dimensional, it can be expressed by a 1-dimensional Poincaré map on the space of Kasner parameters and this approximation gets better as the singularity is approached (Belinskii *et al.*, 1970; Misner, 1969; Barrow, 1982). Another way of expressing this is to use the coarse-grained entropies (2) to give an estimate of the Hausdorff dimension of the space of Kasner configurations (Young, 1982). It should also be mentioned that the normalization of the measures given by N_n is crucial, for otherwise we would obtain the result that the entropy *increases* toward the singularity.

The use of the measures μ_n and μ is unavoidable when analyzing this system, due to the high value of the Kolmogorov entropy associated with the underlying classical system: after a few iterations of T the system is totally chaotic and only a statistical description is possible. Thus, the Kasner wave function $|z\rangle$ alone does not give a complete specification of the model and a density matrix approach provides a better setting for our discussion. The disorder associated with the measures μ_n decreases toward the singularity and we conclude that this cannot be the natural direction of evolution. Consequently, an inference can be made: if this system has an arrow of time, then it cannot point in the direction of contraction of the universe.

Three final observations are in order. (1) If the form of $\mu_n(z)$ discussed above is not used, then nothing can be concluded about the entropy, (2) although we have not proved this system has an arrow of time, we have found a necessary condition for its existence, and (3) the evolution of the Mixmaster canonical variables (Belinskii *et al.*, 1970; Misner, 1969; Barrow 1982) can be described by either time parameters $\tau_1 = \ln g$ or $\tau_2 = -\ln g$, $g = \det g_{ij}$, g_{ij} the spatial metric tensor. Then the singularity is approached as $\tau_1 \to -\infty$ or $\tau_2 \to +\infty$ and there is no contradiction here with the possible existence of an arrow of time, since τ_1 and τ_2 are mere parameters and in both cases the universe contracts.

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